Topology on Projective Space

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ABSTRACT

From a build a topology on projective space, we define some properties of this space.

In projective geometry, a hyper quadric is the set of points of a projective space where a certain quadratic form on the homogeneous coordinates becomes zero.

Let P be a projective space how could we introduce a good topology from oval hyper quadratic, which can establish the properties of the resulting topological space: points, lines, ... and classification of subsets of P.

Keywords: Projective space, Oval hyper quadric, Topology, Homogeneous coordinates

1 INTRODUCTION

In mathematics, a projective space can be thought of as the set of lines through the origin of a vector space V. The cases when $V = \mathbb{R}^2$ and $V = \mathbb{R}^3$ are the real projective line and the real projective plane, respectively, where $\mathbb{R}$ denotes the field of real numbers, $\mathbb{R}^2$ denotes ordered pairs of real numbers, and $\mathbb{R}^3$ denotes ordered triplets of real numbers.

The idea of a projective space relates to perspective, more precisely to the way an eye or a camera projects a $3D$ scene to a $2D$ image. All points that lie on a projection line (i.e., a "line-of-sight"), intersecting with the entrance pupil of the camera, are projected onto a common image point. In this case, the vector space is $\mathbb{R}^3$ with the camera entrance pupil at the origin, and the projective space corresponds to the image points.

Projective spaces can be studied as a separate field in mathematics, but are also used in various applied fields, geometry in particular. Geometric objects, such as points, lines, or planes, can be given a representation as elements in projective spaces based on homogeneous coordinates. As a result, various relations between these objects can be described in a simpler way than is possible without homogeneous coordinates. Furthermore, various statements in geometry can be made more consistent and without exceptions. For example, in the standard Euclidian geometry for the plane, two lines always intersect at a point except when the lines are parallel. In a projective representation of lines and points, however, such an intersection point exists even for parallel lines, and it can be computed in the same way as other intersection points.

Other mathematical fields where projective spaces play a significant role are topology, the theory of Lie groups and algebraic groups, and their representation theories.

Topology can be formally defined as "the study of qualitative properties of certain objects (called topological spaces) that are invariant under a certain kind of transformation (called a continuous map), especially those properties that are invariant under a certain kind of transformation (called homeomorphism)."

Topology is also used to refer to a structure imposed upon a set X, a structure that essentially 'characterizes' the set X as a topological space by taking proper care of properties such as convergence, connectedness and continuity, upon transformation.

Topological spaces show up naturally in almost every branch of mathematics. This has made topology one of the great unifying ideas of mathematics.

The motivating insight behind topology is that some geometric problems depend not on the exact shape of the objects involved, but rather on the way they are put together. For example, the square and the circle have many properties in common: they are both one dimensional objects (from a topological point of view) and both separate the plane into two parts, the part inside and the part outside.

In one of the first papers in topology, Leonhard Euler demonstrated that it was impossible to find a route through the town of Königsberg (now Kaliningrad) that would cross each of its seven bridges exactly once. This result did not depend on the lengths of the bridges, nor on their distance from one another, but only on connectivity properties: which bridges connect to which islands or riverbanks. This problem in introductory mathematics called Seven Bridges of Königsberg led to the branch of mathematics known as graph theory.

The term topology also refers to a specific mathematical idea central to the area of mathematics called topology. Informally, a topology tells how elements of a set relate spatially to each other. The same set can have different topologies. For instance, the real line, the complex plane, and the Cantor set can be thought of as the same set with different topologies.

Formally, let X be a set and let τ be a family of subsets of X. Then τ is called a topology on X if:

1. Both the empty set and X are elements of τ
2. Any union of elements of τ is an element of τ
3. Any intersection of finitely many elements of τ is an element of τ

If τ is a topology on X, then the pair (X, τ) is called a topological space. The notation X_τ may be used to denote a set X endowed with the particular topology τ.

The members of τ are called open sets in X. A subset of X is said to be closed if its complement is in τ (i.e., its complement is open). A subset of X may be open, closed, both (clopen set), or neither. The empty set and X itself are always both closed and open. An open set containing a point x is called a 'neighbourhood' of x.

A set with a topology is called a topological space.

This axiomatization is due to Felix Hausdorff. Let X be a set the elements of X are usually called points, though they can be any mathematical object. We allow X to be empty. Let N be a function assigning to each x (point) in X a non-empty collection N(x) of subsets of X. The elements of N(x) will be called neighbourhoods of x with respect to N (or, simply, neighbourhoods of x). The function N is called a neighbourhood topology if the axioms below [1] are satisfied; and then X with N is called a topological space.

1. If N is a neighbourhood of x (i.e., $N \in N(x)$), then $x \in N$. In other words, each point belongs to every one of its neighbourhoods.
2. If N is a subset of X and contains a neighbourhood of x, then N is a neighbourhood of x. I.e., every superset of a neighbourhood of a point x in X is again a neighbourhood of x.
3. The intersection of two neighbourhoods of x is a neighbourhood of x.

4. Any neighbourhood N of x contains a neighbourhood M of x such that N is a neighbourhood of each point of M.

The first three axioms for neighbourhoods have a clear meaning. The fourth axiom has a very important use in the structure of the theory, that of linking together the neighbourhoods of different points of X.

Projective geometry is less restrictive than either Euclidean geometry or affine geometry. It is an intrinsically non-metrical geometry, whose facts are independent of any metric structure. Under the projective transformations, the incidence structure and the relation of projective harmonic conjugates are preserved. A projective range is the one-dimensional foundation. Projective geometry formalizes one of the central principles of perspective art: that parallel lines meet at infinity, and therefore are drawn that way. In essence, a projective geometry may be thought of as an extension of Euclidean geometry in which the "direction" of each line is subsumed within the line as an extra "point", and in which a "horizon" of directions corresponding to coplanar lines is regarded as a "line". Thus, two parallel lines meet on a horizon line in virtue of their possessing the same direction. [8, 9, 10]

Idealized directions are referred to as points at infinity, while idealized horizons are referred to as lines at infinity. In turn, all these lines lie in the plane at infinity. However, infinity is a metric concept, so a purely projective geometry does not single out any points, lines or plane in this regard-those at infinity are treated just like any others.

In this note, the expression << projective space >> designate a set of points with a straight family subsets called, checking following conditions:

(i) any pair of distinct points p, q is contained in one and only one right denoted p + q;

(ii) any right contains at least three points;

(iii) where a, b, c, d, e are distinct points such that a + b = a + e, and a+ d = a + e, straight b + c and a+ d have a common point.

Various concepts such as linear variety, size, perceptivity, hyper plane and their properties, will be used without reminders. It is well known that projective spaces can be classified in the following manner:

a) trivial projective spaces characterized by fair they have at most a right, or that their size is less than 2;

b) non artesian projective planes;

c) artesian projective spaces of dimension ≥ 2, which are none other than the projective spaces in homogeneous coordinates in a body.

Note that a linear manifold will always be considered as a set of points.

Let P be a projective space with homogeneous coordinates X_i, $i \in I$, with values in

a commutative field K. While hyper quadric of P is the set of points whose coordinates satisfy an equation of the second degree of the form: $\sum_{i,j\in I} Q_{ij} X_i X_j = 0$.

Observe that this definition still has a meaning, even if I is a infinite set.

An oval in a projective or affine space is a point set which has similar properties considering incidence with lines as a circle or ellipse in the real Euclidean plane.[5, 6]

Characterized hyper quadrics the oval by the fact that do not contain any line and at least two points, leading to the notion of ovoid, a generalization due to Tits who showed interest in his discovery by of < ovoïdes de Suzuki) > [3, 4, 7] and various characterizations oval hyper quadric based on the homogeneity of these units [2].

Definition 1:

A non empty point set S of a projective space is called oval if the following properties are fulfilled:

(o1) Any line meets S in at most two points.

(o2) For any point $P \in S$ there is one and only one line D such that $D \cap S = \{P\}$.

A line D is a exterior or tangent or secant line of the oval S if $|D \cap S| = 0$ or $|D \cap S| = 1$ or $|D \cap S| = 2$, respectively.

2 MAIN RESULTS

In this section, P is a projective space with values in a commutative field K.

Definition 2:

Let $X \in P$ and $V \subset P$ with $X \in V$.

V is said neighbourhood of X if only if every oval hyper quadric Q_X via X, the intersection of $Q_X - \{X\}$ with V is different from the empty set.

Remark 1:

The definition of the neighbourhood of X in P has a meaning, the four axioms below [1] are checked.

Let A be a subset of P, we say that A is open if and only if A is a neighbourhood of each of these points.

We define a closed as an open complementary.

Lemma 1:

Let $X \in P$, the set $\{X\}$ is closed in P.

Proof :

Let $A = P - \{X\}$ is the complementary of $\{X\}$, then A is open in P.

Let $Y \in P$, $Y \neq X$ and Q_Y is the oval hyper quadric via Y.

We have :

$$(Q_Y - \{Y\}) \cap A = Q_Y - \{X, Y\} \neq \emptyset$$

So, the set $\{X\}$ is closed in P.

Lemma 2:

Every line in P is closed.

Proof :

Let D be a line in P. Then, for every point X in D, there is an oval hyper quadric Q_X via X such that D tangent to Q_X in X.

So,

$$(Q_X - \{X\}) \cap D = \emptyset$$

whence , D is closed.

Lemma 3:

Let $X \in P$ and V is neighbourhood of X.

There exists $\epsilon > 0$ such that $B(X, \epsilon) \subset V$.

Proof :

Let V is neighbourhood of X. Then for every oval hyper quadric Q_X via X we have :

$$(Q_X - \{X\}) \cap V \neq \emptyset$$

Let $(Y_i)_{i \in I}$ is a sequence of elements of $(Q_X - \{X\}) \cap V$.

We consider $\epsilon_i = d(X, Y_i)$, we take $\epsilon = \underbrace{\min}_{i \in I} \epsilon_i$, we have the ball $B(X, \epsilon) \subset V$.

Theorem 1:

Every projective subspace of P is closed.

Proof :

It suffices to prove that any hyper plane of P is closed.

Let H be an hyper plane of P, then $\overline{H} = P - H$.

Let $X \in \overline{H}$ then $(Q_X - \{X\}) \cap \overline{H} \neq \emptyset$ because if not, there will be $\overline{H}$ is tangent to Q_X in X, which contradit that $X \in \overline{H}$.

so , H is closed.

Like any projective subspace is seen as intersecting of hyper plane, we deduct the result of the theorem.

Theorem 2:

Every hyper quadric of P is closed.

Proof :

Let Qis an hyper quadric of P, then $\overline{Q} = P - Q$.

Let $X \in \overline{Q}$ and Q_X is an oval hyper quadric via X. We have :

$$(Q_X - \{X\}) \cap \overline{Q} = (Q_X - Q) - \{X\} \neq \emptyset$$

So, Q is closed.

Remark 2:

- Every projective subspace of P is compact.
- Every hyper quadric of P is compact.

3 CONCLUSION

In this paper we introduce a new definition of topology on projective space, we proof that every hyper quadric of P is closed. I think we can use this new topological concept for show a lot of results, this seems difficult, but simple, using this new topology, just defined.

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